UDC 536.248

UNSTEADY-STATE RADIATIVE HEAT TRANSFER BETWEEN TWO OPAQUE BODIES OF FINITE DIMENSIONS

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Inzhenerno-Fizicheskii Zhurnal, Vol. 8, No. 6, pp. 773-778, 1965

The unsteady-state radiative interaction of two opaque gray bodies according to the Stefan-Boltzmann law is considered. The problem is reduced to a non-linear Volterra integral equation for the net radiation density. A solution is obtained for a linear approximation. The general case and the short-time behavior are discussed.

The thermal interaction between heat-conducting bodies which are insulated from external influences can be reduced to unsteady-state heat transfer between their surfaces which results in the establishment of temperature fields. Depending on the physical and optical properties of the interacting bodies, as well as on their surface properties, one can distinguish between perfect and "imperfect" thermal contacts. The latter include all thermal contacts occuring in nature, in which the interaction takes place by conduction, convection, or radiation.

Recently, the interest in these problems has increased due to the increasing number of investigations of fast processes.

An essential condition, which makes it possible to formulate these problems of thermal interaction, is the commensurability of the resulting heat fluxes in the interacting bodies at corresponding moments of time.

These conditions can be realized during the initial period of interaction in heat transfer between arbitrary bodies, as well as in the case when the interacting bodies have thermal capacities of the same order [1]. When the interacting bodies have finite thermal conductivity (finite dimensions), the problem becomes complicated due to the effect of heat conduction from the surfaces.

In the following we shall consider the radiative interaction between two plane-parallel infinite plates of finite thickness which have different thermophysical properties. The interacting surfaces are in thermal radiative contact. External thermal influences are excluded.

The problem may be formulated as follows:

$$\frac{\partial^2 T_1(x,\tau)}{\partial x^2} = \frac{1}{a_1}\frac{\partial T_1(x,\tau)}{\partial \tau}, \quad -R \leqslant x \leqslant 0,$$

$$\frac{\partial^2 T_2(x,\tau)}{\partial x^2} = \frac{1}{a_2}\frac{\partial T_2(x,\tau)}{\partial \tau}, \quad 0 \leqslant x \leqslant R, \tag{1}$$

$$\lambda_1 \left.\frac{\partial T_1}{\partial x}\right|_{x=-R} = 0, \quad \lambda_1 \left.\frac{\partial T_1}{\partial x}\right|_{x=0} = \sigma_{12}\,[T_2^4(0,\tau) - T_1^4(0,\tau)] = E_1(0,\tau),$$

$$\lambda_2 \left.\frac{\partial T_2}{\partial x}\right|_{x=R} = 0, \quad -\lambda_2 \left.\frac{\partial T_2}{\partial x}\right|_{x=0} = \sigma_{21}\,[T_1^4(0,\tau) - T_2^4(0,\tau)] = E_2(0,\tau), \tag{2}$$

$$T_1(x,0) = T_1, \quad T_2(x,0) = T_2. \tag{3}$$

The general solution in transform form is

$$\bar{u}_1(x,p) = \frac{T_1}{p} + \frac{\bar{E}_1(0,p)}{\lambda_1\sqrt{p/a_1}}\;\frac{\operatorname{ch}\sqrt{p/a_1}\,(x+R)}{\operatorname{sh}\sqrt{p/a_1}\,R},$$

$$\bar{u}_2(x,p) = \frac{T_2}{p} + \frac{\bar{E}_2(0,p)}{\lambda_2\sqrt{p/a_2}}\;\frac{\operatorname{ch}\sqrt{p/a_2}\,(R-x)}{\operatorname{sh}\sqrt{p/a_2}\,R}. \tag{4}$$

Here

$$\bar{u}_i(x, p) = \int_0^\infty T_i(x, \tau)\,[\exp(-p\tau)]\,d\tau,$$

$$\bar{E}_i(0, p) = \int_0^\infty E_i(0, \tau)\,[\exp(-p\tau)]\,d\tau \quad (i = 1, 2).$$

The expressions for the temperatures $T_i(0, \tau)$ of the interacting surfaces, obtained by the inverse Laplace transformation, are

$$T_i(0, \tau) = T_i + \frac{a_i}{\lambda_i R}\int_0^\tau \left[1 + 2\sum_{n=1}^\infty \exp[-\mu_i n^2(\tau - t)]\right]\cdot E_i(0, t)\,dt. \tag{5}$$

Here

$$\mu_i = \pi^2 a_i/R^2 \quad (i = 1, 2).$$

Taking into account that $E_1(0, \tau) = -E_2(0, \tau)$ and using (5), we obtain the solution for the net radiation density

$$\begin{aligned} E_1(0, \tau) = \sigma_{12}\Bigg\{&\left(T_2 - \frac{a_2}{\lambda_2 R}\int_0^\tau \left[1 + 2\sum_{n=1}^\infty \exp[-\mu_2 n^2(\tau - t)]\right] E_1(0, t)\,dt\right)^4 - \\ &-\left(T_1 + \frac{a_1}{\lambda_1 R}\int_0^\tau \left[1 + 2\sum_{n=1}^\infty \exp[-\mu_1 n^2(\tau - t)]\right] E_1(0, t)\,dt\right)^4\Bigg\}, \end{aligned} \tag{6}$$

which can be represented also in the form

$$\begin{aligned} \varphi(\tau) = \delta_2 &\left\{1 - \eta_2\int_0^\tau \left[1 + 2\sum_{n=1}^\infty \exp[-\mu_2 n^2(\tau - t)]\right] \varphi(t)\,dt\right\}^4 - \\ -\delta_1 &\left\{1 + \eta_1\int_0^\tau \left[1 + 2\sum_{n=1}^\infty \exp[-\mu_1 n^2(\tau - t)]\right] \varphi(t)\,dt\right\}^4, \end{aligned} \tag{7}$$

where

$$\varphi(\tau) = E_1(0, \tau)/\sigma_{12}(T_2^4 - T_1^4), \quad \delta_1 = T_1^4/(T_2^4 - T_1^4), \quad \delta_2 = T_2^4/(T_2^4 - T_1^4);$$

$$\eta_1 = \frac{a_1\sigma_{12}}{\lambda_1 R}\,\frac{T_2^4 - T_1^4}{T_1}, \quad \eta_2 = \frac{a_2\sigma_{12}}{\lambda_2 R}\,\frac{T_2^4 - T_1^4}{T_2}.$$

Equation (7) shows that the process of radiative heat transfer between the bodies under consideration is described by a non-linear Volterra integral equation for the dimensionless net radiation density $\varphi(\tau)$ and, consequently, requires deeper discussion.

In the following we shall present an approximate solution of equation (7) by the method of successive approximations, using a linearization described in [1].

Equation (7), linearized by the above method, can be rewritten in the form

$$\varphi(\tau) = \delta_2\{4[1 - \eta_2\varphi_2(\tau)]\,z_2^3\} - \delta_1\{4[1 + \eta_1\varphi_1(\tau)]\,z_1^3\} - 3(\delta_2 z_2^4 - \delta_1 z_1^4), \tag{8}$$

where

$$z_1 = 1 + \eta_1\varphi[(k-1)\Delta\tau]\int_{(k-1)\Delta\tau}^{k\Delta\tau} \left[1 + 2\sum_{n=1}^\infty \exp[-\mu_1 n^2(k\Delta\tau - t)]\right] dt,$$

$$z_2 = 1 - \eta_2\varphi[(k-1)\Delta\tau]\int_{(k-1)\Delta\tau}^{k\Delta\tau} \left[1 + 2\sum_{n=1}^\infty \exp[-\mu_2 n^2(k\Delta\tau - t)]\right] dt;$$

$$\varphi_i(\tau) = \int_0^{\tau} \left[1 + 2 \sum_{n=1}^{\infty} \exp\left[-\mu_i n^2 (\tau - t)\right] \right] \varphi(t)\, dt \quad (i = 1, 2).$$

After several transformations, equation (8) can be brought to the form

$$\begin{gathered} \varphi(\tau) = f(z_i) + \lambda \int_0^{\tau} K(\tau - t)\, \varphi(t)\, dt, \\ f(z_i) = 4\left(\delta_2 z_2^3 - \delta_1 z_1^3\right) - 3\left(\delta_2 z_2^4 - \delta_1 z_1^4\right), \ \lambda = 4\left(\delta_1 \eta_1 z_1^3 + \delta_2 \eta_2 z_2^3\right), \\ -K(\tau) = 1 + \left[\delta_1 \eta_1 z_1^3 \sum_{n=1}^{\infty} \exp\left(-\mu_1 n^2 \tau\right) - \delta_2 \eta_2 z_2^3 \sum_{n=1}^{\infty} \exp\left(-\mu_2 n^2 \tau\right)\right] \times \\ \times \left[2\left(\delta_1 \eta_1 z_1^3 + \delta_2 \eta_2 z_2^3\right)\right]^{-1}. \end{gathered} \tag{9}$$

The kernel $K(\tau)$ of equation (9) satisfies additional conditions [2], which make it possible to obtain a closed-form solution.

Applying the Laplace transform to equation (9) and to its solution

$$\varphi(\tau) = f(z_i) + \lambda \int_0^{\tau} R(\tau - t)\, f(z_i)\, dt, \tag{10}$$

we obtain the expression $M(p)$ for the transform of the resolvent $R(\tau - t)$ of equation (9) in the form

$$\begin{gathered} M(p) = -\left(\gamma_1 \frac{\operatorname{cth} \sqrt{p/a_1}\, R}{\sqrt{p/a_1}\, R} + \gamma_2 \frac{\operatorname{cth} \sqrt{p/a_2}\, R}{\sqrt{p/a_2}\, R}\right) \times \\ \times \left(1 + \gamma_1 \frac{\operatorname{cth} \sqrt{p/a_1}\, R}{\sqrt{p/a_1}\, R} + \gamma_2 \frac{\operatorname{cth} \sqrt{p/a_2}\, R}{\sqrt{p/a_2}\, R}\right)^{-1}, \end{gathered} \tag{11}$$

where

$$\gamma_i = 4\delta_i \eta_i z_i^3 R^2 / a_i \quad (i = 1, 2).$$

Applying the inverse Laplace transform

$$R(\tau) = \frac{1}{2\pi i} \int_{\varepsilon - i\infty}^{\varepsilon + i\infty} M(p) \exp(p\tau)\, dp,$$

and using some simple transformations, the expression for the resolvent of equation (9) can be reduced to the form

$$R(\tau) = -2 \frac{a_1}{R^2} \nu_m^2 \exp\left(-\nu_m^2 \frac{a_1}{R^2} \tau\right) \left(1 + \frac{\gamma_1}{\sin^2 \nu_m} + \frac{\gamma_2 \sqrt{a_1/a_2}}{\sin^2 \sqrt{a_1/a_2}\, \nu_m}\right)^{-1}. \tag{12}$$

The expression for the dimensionless net radiation density is, in its final form,

$$\begin{gathered} \varphi(\tau) = f(z_i) \left[1 - 2 \sum_{m=1}^{\infty} \left[1 - \exp\left(-\nu_m^2 \tau a_1 / R^2\right)\right] \times \right. \\ \left. \times \left[1 + \frac{\gamma_1}{\sin^2 \nu_m} + \frac{\gamma_2 \sqrt{a_1/a_2}}{\sin^2 \sqrt{a_1/a_2}\, \nu_m}\right]^{-1}\right]. \end{gathered} \tag{13}$$

where ν_m are the roots of the transcendental equation

$$\gamma_1 \operatorname{ctg} \nu_m + \gamma_2 \operatorname{ctg} \sqrt{a_1/a_2}\, \nu_m = \nu_m. \tag{13'}$$

In the range of low Fourier numbers ($Fo_1 = a_1\tau/R^2$), the dimensionless group $\sqrt{p/a_i}\, R$ takes on high values [3], which makes it possible to introduce in equation (11) considerable simplifications, so that

$$M(p) \approx -\omega/(\sqrt{p} + \omega), \tag{14}$$

where

$$\omega = R^{-1}(\gamma_1 \sqrt{a_1} + \gamma_2 \sqrt{a_2}).$$

Inverting (14), we obtain the resolvent

$$R(\tau) = \omega \left[\frac{1}{\sqrt{\pi\tau}} - \omega \exp(\omega^2\tau)(1 - \operatorname{erf} \omega\sqrt{\tau}) \right]. \tag{15}$$

For small values of τ we can use the series

$$\operatorname{erf} \omega\sqrt{\tau} = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{\omega^{2n+1}\tau^{n+1/2}}{n!(2n+1)}, \tag{16}$$

and, taking account of (15) and (16), we can bring equation (10) to the form

$$\varphi(\tau) = f(z_i)\left[\exp \omega^2\tau - \frac{2\omega}{\sqrt{\pi}}\sqrt{\tau} - \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{\omega^{2n+3}}{n!(2n+1)} \times \right.$$
$$\left. \times \int_0^{\tau} \exp[\omega^2(\tau - t)](\tau - t)^{n+1/2} dt \right]. \tag{17}$$

Expanding $\exp \omega^2\tau$ in a series and taking the first two terms, we obtain the final form of the short-time solution

$$\varphi(\tau) \approx f(z_i)\left\{ \exp \omega^2\tau - \frac{2\omega}{\sqrt{\pi}}\sqrt{\tau} - \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \times \right.$$
$$\left. \times \frac{\omega^{2n+3}}{n!(2n+1)} \times \left[\frac{\tau^{n+3/2}}{n+3/2} + \omega^2 \frac{\tau^{n+5/2}}{n+5/2} \right] \right\}. \tag{18}$$

Note that in the case of thermal interaction according to Newton's law (unsteady-state interaction between a boundary layer and a body surface, when the thermal capacities of the body and of the boundary layer are of the same order), the solution can be significantly simplified. In particular, the expression for the dimensionless net heat flux becomes

$$\varphi(\tau) = 1 - 2\sum_{m=1}^{\infty} [1 - \exp(-\nu_m^2 \tau a_1/R^2)] \times$$
$$\times \left[1 + \frac{\gamma_1}{\sin^2 \nu_m} + \frac{\gamma_2\sqrt{a_1/a_2}}{\sin\sqrt{a_1/a_2}\,\nu_m} \right]^{-1},$$

where

$$\gamma_i = \alpha a_i/\lambda_i R, \quad \varphi(\tau) = [T_2(0, \tau) - T_1(0, \tau)]/[T_2 - T_1].$$

The figure shows the dimensionless net radiation density as a function of time for the case of interaction between the following plates: $R_1 = R_2 = 0.1$ m; $a_1 = \lambda_1/c_1\rho_1 = 0.612$ m^2/hr, $T_1 = 300$°K, $a_2 = \lambda_2/c_2\rho_2 = 0.0636$ m^2/hr, $T_2 = 1500$°K.

In the course of the calculations it was found that the value of $f(z_i)$ does not differ significantly from unity (the value of $f(z_i)$ at $\tau = 0$), which makes the calculations considerably simpler.

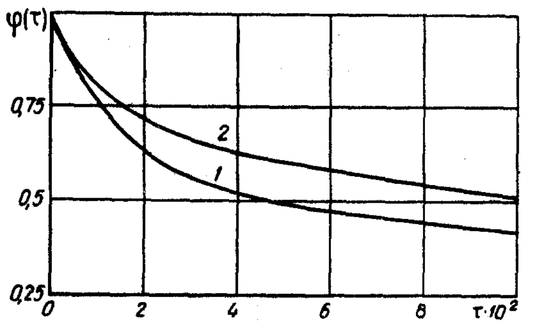
The evaluation of the roots of the transcendental equation (13) reduces to the evaluation of the root ν_1 only, which makes the greatest contribution to the variation of $\varphi(\tau)$ as a function of τ.

The figure also shows the variation of $\varphi(\tau)$ for the case of two semi-infinite bodies with the same thermophysical properties as listed above. Note that the initial parts of the curves coincide only for very small values of τ.

NOTATION

a_i – thermal diffusivities; $E_i(0, \tau)$ – net radiation densities; $\sigma_{12} = \sigma_{21} = \left(\frac{1}{A_1} + \frac{1}{A_2} - 1\right)^{-1}$ – effective emissivity;

A_1, A_2 – absorptivities of the interacting bodies; λ_i – coefficients of thermal conductivity; z_1, z_2 – values of functionals, involving dimensionless net fluxes averaged over $\Delta\tau$, which are constant over $\Delta\tau$; α – coefficient of convective heat transfer.



Dimensionless net radiation density between two bodies $\varphi(\tau)$ as a function of time τ (hr): 1) According to Eq. (13); 2) solution for semi-infinite bodies [1].

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18 June 1964

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